Abstract

In traditional voting environments, “winner takes all”, and there is no compensation for agents who voted for loosing candidates. This tyranny of majority may be remedied by introducing “random voting” procedures. Agents cast their vote, and based on it a distribution over candidates is chosen.

It may be a real randomization (we toss a coin to pick who does an unpleasant task), or a time-share (each of us gets to enjoy a prize a fraction of time). Or it could be a “fractional” outcome, specifying budget shares in budget to different projects, or in number of sits in a governing body allocated to different parties.

In addition, when we allow for random vote, Condorcet paradox disappears. We present the maximal lottery rule that always select Condorcet winner, and discuss its properties.

Random Voting

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Candidates (alternatives): $A = \{a_1, ..., a_k\}$

Voters: $N = \{1, ..., n\}$

Preferences of $i \in N$: a strict order $\succ_i$ or R_i (permutation) of A, i.e.:

$$a_{i(1)} \succ_i a_{i(2)} \succ_i a_{i(3)} \succ_i ... \succ_i a_{i(k-1)} \succ_i a_{i(k)}$$

A "preference profile": $R = (R_1, ..., R_n)$

Voting rule: $f : R = (R_1, ..., R_n) \longmapsto a \in A$

Random voting rule: $F : (R_1, ..., R_n) \longmapsto p \in \Delta$

$\Delta = \Delta(A)$ is the set of all lotteries/distributions on A:

$$\Delta = \left\{p = (p_1, ..., p_k) \in \mathbb{R}^k : \sum_{a=1}^{k} p_a = 1, \text{ all } p_a \geq 0\right\}$$

Random Voting:

Random outcome: $p = (p_1, ..., p_k) \in \Delta(A)$

Meaning: candidate $a \in A$ is to be chosen with the probability p_a

Alternative interpretations: each candidate $a \in A$ is getting p_a fraction of the total resource we divide (time, money..); proportional representation

Allows to avoid "tyranny of majority", to guarantee anonymity/neutrality

Reasonable fairness requirements: minority guarantees. Examples:

Any $i \in N$ should get at least $\frac{1}{n}$-th chance of her best outcome

Any group $V \subset N$ should have power over at least $\frac{|V|}{n}$ fraction of the resources

Random Dictator:

f^{RD} selects the best candidate for each agent i with probability $1/n$

Is anonymous and neutral, and gives fair shares to minorities.

It is strategy-proof (it is never helpful to lie about one's preferences)

It fails to reach a fair compromise when opinions are polarized (and so may be very inefficient)

It fails to select a Condorcet winner if it exists

Dichotomous (0-1) Preferences:

Each $i \in N$ "approves" a subset $S_i \subset A$ – "good candidates"

Some Rules:

Utilitarian: maximizes the sum of agents' shares of good candidates $\sum_i p(S_i)$

Nash: maximizes the product of agents' shares of good candidates $\prod_i p(S_i)$

Egalitarian: maximizes the share of least satisfied agent, then 2nd worst, etc.

Theorem: no rule satisfies Efficiency, Strategy-Proofness, and Positive Share

Efficiency (Eff): there is no way to improve everybody
Strategy-proofness (SP): it is never helpful to report $S' \neq S_i$
Positive share (PS): each $i \in N$ gets positive share for some $a \in S_i$

Easy variant: no rule is anonymous, neutral, Eff, SP, and PS

Proof: for $n = 5$, $A = \{a, b, c, d\}$

Let $R = (S_1, S_2, S_3, S_4, S_5) = (\{a, c\}, \{a, d\}, \{b, c\}, \{a, b\}, \{a\})$
Let $f(R) = \delta = (\delta(a), \delta(b), \delta(c), \delta(d))$
By symmetry, $\delta(b) = \delta(c)$; by PS for agent 3 $\delta(b) = \delta(c) > 0$
Hence, share of agent 4 is at most $1 - \delta(c) < 1$

Suppose agent 4 reports instead $S'_4 = \{b, d\}$,
so $R' = (\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a\})$ and $\delta' = f(R')$
By symmetry $\delta'(c) = \delta'(d) = \varepsilon$
If $\varepsilon > 0$ then let $p(c) = p(d) = 0$, $p(a) = \delta'(a) + \varepsilon$, $p(b) = \delta'(b) + \varepsilon$
p is preferred to δ' by all agents
If $\varepsilon = 0$ then share of agent 4 is $\delta'(a) + \delta'(b) = 1$

Majority vote between pairs of candidates:

Given R and $a, b \in A$, let $R(a,b) = |\{i \in N : a \succ_i b\}| - |\{i \in N : b \succ_i a\}|$

$R(a,b)$ is a "gain" (majority margin) of a over b under preferences R
Note: $R(b,a) = -R(a,b)$

Condorcet Winner: $a \in A$ such that $R(a,c) \geq 0$ for all $c \in A$, $c \neq a$

Under classical "deterministic" voting may fail to exist!

Example ("Condorcet cycle"): $A = \{a, b, c\}$, $N = \{1, 2, 3\}$
Preferences are $1 : a \succ_1 b \succ_1 c$; $2 : b \succ_2 c \succ_2 a$; $3 : c \succ_3 a \succ_3 b$
$R(a,b) = R(b,c) = R(c,a) = 1$, $R(b,a) = R(c,b) = R(a,c) = -1$

What about random voting?

Examples

$$\begin{matrix} 1 & 1 & 1 \\ a & b & c \\ b & a & a \\ c & c & b \end{matrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{matrix} \\ a \\ b \\ c \end{matrix} \begin{matrix} a & b & c \\ \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \end{matrix} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \geq 0$$

$$\begin{matrix} 1 & 1 & 1 \\ a & b & c \\ b & c & a \\ c & a & b \end{matrix} \qquad \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}^T \begin{matrix} \\ a \\ b \\ c \end{matrix} \begin{matrix} a & b & c \\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{matrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \geq 0$$

Some matrix algebra

Let $M = \left(m_{ij}\right)_{1 \leq i,j \leq k}$ be k by k square matrix

Let $p, q \in \mathbf{\Delta} = \left\{p = (p_1, ..., p_k) \in \mathbb{R}^k : \sum_{r=1}^{k} p_r = 1, \text{ for all } p_r \geq 0\right\}$

Think about p, q as two independent lotteries (distributions) over $\{1, ..., k\}$
Suppose we choose a row of matrix M using p, and a column of M using q
Then an ij-entry m_{ij} of M is chosen with the probability $p_i q_j$

The expected value of the matrix' entry m chosen is thus

$$pMq^T = (p_1, ..., p_k) \begin{pmatrix} m_{11} & ... & m_{1k} \\ \\ m_{k1} & ... & m_{kk} \end{pmatrix} \begin{pmatrix} q_1 \\ ... \\ q_k \end{pmatrix} = \sum_{1 \leq i,j \leq k} p_i m_{ij} q_j$$

Condorcet Winner: Random Voting

Fix R. Let $p, q \in \Delta(A)$ be two random outcomes, run independently

Let $M = \left(m_{ij}\right)_{1 \le i,j \le k} = \left(R(a_i, a_j)\right)_{1 \le i,j \le k}$ be matrix of gains of rows over columns

Each event [p chooses a_i and q chooses a_j] has probability $p_i q_j$

In this event, gain of p over q is $m_{ij} = R(a_i, a_j)$

Thus, the expected majority gain of p over q is pMq^T

Condorcet Winner: a random outcome p^* s.t. $p^* M q^T \ge 0$ for all $q \in \Delta(A)$

Fact: Condorcet Winner always exists in random voting

MiniMax Theorem

Given any matrix $M = \left(m_{ij}\right)$ $(1 \leq i, j \leq k)$

$$\max_{x\in\Delta}\min_{y\in\Delta} xMy^T = \min_{y\in\Delta}\max_{x\in\Delta} xMy^T = v = v(M)$$

If M is skew-symmetric: $M = -M^T$, then $v = 0$.

Now: $M = \left(R(a_i, a_j)\right)_{1\leq i,j\leq k}$ is skew symmetric: $m_{ji} = -m_{ij}$, so $v = 0$

Let $y^*(x) \in \arg\min_{y\in\Delta} xMy^T$; $p^* \in \arg\max_{x\in\Delta}\left(\min_{y\in\Delta} xMy^T\right) = \arg\max_{x\in\Delta}\left(xM\left(y^*(x)\right)^T\right)$

We have $p^*Mq^T \geq p^*M\left(y^*(p^*)\right)^T = v = 0$ for any $q \in \Delta$

Lemma: $\max_{x\in\Delta}\min_{y\in\Delta} xMy^T \le \min_{y\in\Delta}\max_{x\in\Delta} xMy^T$

Proof: let $p^* \in \arg\max_{x\in\Delta}\left(\min_{y\in\Delta} xMy^T\right)$ and $q^* \in \arg\min_{y\in\Delta}\left(\max_{x\in\Delta} xMy^T\right)$

$$\max_{x\in\Delta}\min_{y\in\Delta} xMy^T = \min_{y\in\Delta} p^*My^T \le p^*M(q^*)^T \le \max_{x\in\Delta} xM(q^*)^T = \min_{y\in\Delta}\max_{x\in\Delta} xMy^T$$

Lemma: If M is skew-symmetric: $M = -M^T$, or $m_{ji} = -m_{ij}$, then $v = 0$

Proof: xMy^T is a number, so $xMy^T = \left(xMy^T\right)^T = -yMx^T$

Next, $\max_x \min_y (-g(x,y)) = \max_x \left(-\max_y g(x,y)\right) = -\min_x \max_y g(x,y)$; so

$$v = \max_{x\in\Delta}\min_{y\in\Delta} xMy^T = \max_{x\in\Delta}\min_{y\in\Delta}\left(-yMx^T\right) = -\min_{x\in\Delta}\max_{y\in\Delta} yMx^T = -v$$

MiniMax Theorem: $\max_{x\in\Delta}\min_{y\in\Delta} xMy^T \geq \min_{y\in\Delta}\max_{x\in\Delta} xMy^T$

Proof by separation argument:

Let $S = \{My^T : y \in \Delta\}$,

$T_w = \{z \in \mathbb{R}^k : \text{all } z_a \leq w\} = \{z \in \mathbb{R}^n : x \cdot z \leq w \text{ for all } x \in \Delta\}$.

$S, T \subset \mathbb{R}^k$, convex, closed; S is bounded, $S \cap T_w = \emptyset$ for w small enough

Let $\overline{v} = \min_w\{w : S \cap T_w \neq \emptyset\}$, $\overline{z} \in S \cap T_{\overline{v}}$. These S and $T_{\overline{v}}$ can be separated by a hyperplane $H = \{z \in \mathbb{R}^n : \overline{x} \cdot z = c\}$, where $\sum_{i=1}^n \overline{x}_i = 1$.

One can check that $c = \overline{v}$ (since $(\overline{v}, ..., \overline{v}) \in H$), and that

$$\max_{x\in\Delta}\min_{y\in\Delta} x \cdot My^T \geq \min_{y\in\Delta} \overline{x} \cdot My^T \geq c = \overline{v} \geq \max_{x\in\Delta} x \cdot \overline{z} \geq \min_{y\in\Delta}\max_{x\in\Delta} x \cdot My^T$$

Note: we also get $v = \overline{v}$, and $p^* = \overline{x}$

Maximal Lotteries (p^*):

– Always exist (Minimax theorem)

– Almost always unique:

set of profiles with multiple p^* has measure zero

unique for odd n, and strict preferences

– Preferences need not be strict, transitive, acyclic, complete..

Characterization Theorem:

Maximal lottery is the only rule that satisfies "Population Consistency" and "Composition Consistency"

Population Consistency: Whenever two disjoint electorates agree on a lottery, this lottery should also be chosen by the union of both electorates

Composition Consistency: Decomposable preference profiles are treated component-wise. In particular, alternatives are not affected by the cloning of other alternatives