# Mathématiques et Democratie

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**Definition**

Given a set $A$ of $p$ candidates $a, b, c, \cdots$, we call *preference* a strict ranking of all the candidates

Given a set $N$ of $n$ voters $i, j, \cdots$, a $N$-profile of preferences on $A$ is a list of $n$ preferences, one for each voter

Each voter casts a ballot reporting some (not necessarily their) complete preferences on $A$

A *voting rule* takes a profile ($N$-vector) of ballots and outputs (elects) a candidate $a$: the winner

choosing a voting rule: the debate between *Borda's scoring rule* and *Condorcet's majority voting* (Paris 1780-s)

*the Borda rule*: a ballot gives 0 points to the last ranked candidate, 1 point to the next to last, and so on; the candidate with the largest total score wins (with some tie-breaking rule)

*general scoring rules*: fix a sequence of weights: $s_0 \leq s_1 \leq \cdots \leq s_{p-1}$ s.t. $s_0 < s_{p-1}$; assign $s_0$ points to the last ranked candidate, $s_1$ to the next to last, and so on

the familiar *plurality voting rule:* $s_0 = s_1 = \cdots = s_{p-2} = 0, s_{p-1} = 1$

**interpretation**: *virtual utilitarianism:* it is *as if* utilities for the different candidates follow the benchmark scale and the winner maximises the sum of individual welfares, i. e., society's welfare

*example 1*: Borda rejects the plurality winner

| 6 | 7 | 8 |
|---|---|---|
| $b$ | $c$ | $a$ |
| $c$ | $b$ | $b$ |
| $a$ | $a$ | $c$ |

so does Condorcet, but for different reasons

Condorcet's rule (aka the *majority rule*):
for any two candidates $a, b$ determine the majority relation, $a \succ_{maj} b$, or $b \succ_{maj} a$, or $a \sim_{maj} b$, by counting how many voters prefer $a$ to $b$ versus $b$ to $a$; the overall winner is the candidate who beats all others: the *Condorcet winner*

**interpretation**: the voters exercise *virtual rights:* the Condorcet winner implements the "will of the majority" (Rousseau: la volonté générale)
the Condorcet winner relies on different majorities to "defeat" other outcomes: it is a power equilibrium; it has no welfarist meaning

*example 2:* the Condorcet and plurality rules agree, and disagree with the choice of the Borda rule

| 15 | 11 |
|---|---|
| $a$ | $b$ |
| $b$ | $c$ |
| $c$ | $a$ |

- the majority rule is prima facie more attractive than Borda's rule because it does not rely on an arbitrary cardinalisation of preferences (the scores): deciding $a$ or $b$ by a simple majority vote is *compelling* (**May's Theorem**)

- but a Condorcet winner does not always exist because of majority cycles and when this happens breaking the cycles of the majority relation is problematic and leads to ad hoc compromises

cyclical majorities: a simple instance

| $n_1$ | $n_2$ | $n_3$ |
|---|---|---|
| $a$ | $c$ | $b$ |
| $b$ | $a$ | $c$ |
| $c$ | $b$ | $a$ |

where each $n_i$ is a strict minority

Condorcet proposed to break the cycle at its weakest link: elect $b$ if $n_3 > n_1, n_2$
but this is not well defined with four or more candidates

the **reunion paradox** is a strong argument *in support* of scoring rules and *against* the majority rule

*Reinforcement axiom*: if two *disjoint* committees $N_1$ and $N_2$ use the same rule $\rho$ and choose the same winner $a$ from the (common) set $A$ of candidates, then the reunion of these two committees using $\rho$ should confirm the election of $a$

this holds for the majority rule *if all goes well*: if $a$ is the Condorcet winner in each committee
however if the majority relation in at least one of the committees is cyclical the reunion of all voters may end up electing another candidate $b$

**Definition:** *Condorcet Consistent voting rule*: one that selects the Condorcet winner when there is one

**Proposition:** (Young [1978]) a Condorcet Consistent voting rule must violate the Reinforcement axiom at some profile of preferences

note: the known proof of statement $i$) requires 13 voters or more
open question: what is the smallest number of voters for which the statement holds?

**Theorem** (Young [1975]) the (large!) family of scoring rules is characterised by the combination of four axioms

Reinforcement, Anonymity, Neutrality and Continuity

*Anonymity:* implements "one person one vote"

*Neutrality:* rules out *a priori* discrimination between the candidates

*Continuity*: if committee $N_1$ elects $a$ and disjoint committee $N_2$ elects $b, b \neq a$, then for $m$ large enough the committee $\underbrace{N_1, N_1, \cdots, N_1}_{m} \cup N_2$ elects $a$

note: the proof is difficult, it relies on the separation theorem

the **strategic properties** of the majority rule contrast favourably with those of the Borda rule (as already noted by Condorcet)

$\longrightarrow$ in **example 1** under the plurality rule the 7 voters $c \succ b \succ a$ ensure the election of $b$ by reporting $b$ as their top choice

$\longrightarrow$ in **example 2** under the Borda rule the 15 voters $a \succ b \succ c$ make $a$ the winner by reporting $a \succ c \succ b$

a voter's *misreport* to the voting rule is a ballot different from their true preference ordering

**Definition**: *strategyproof voting rule*: one where neither of the two following situations can occur:
a single voter benefits from mis-reporting their preference
some larger subset of voters can coordinate the misreports so that each of them benefits

**Lemma**:
$i$) plurality, Borda, and all scoring rules are not strategyproof
$ii$) whenever the Condorcet winner is guaranteed to exist, it defines a strategyproof voting rule

*single-peaked preferences*: the most important domain of preferences where a Condorcet winner always exists

the candidates/*outcomes* are ranked along a fixed scale: numerical quantity (cost, temperature, weight), ideological position from left to right, location frome North to South along a road etc..

$$a_1 < a_2 < \cdots < a_p$$

voter $i$'s preferences are *single-peaked* if their favorite outcome is $a_{i^*}$ (the peak) and

for any $i, j$ s. t. $i < j < i^*$ they strictly prefers $a_j$ to $a_i$

for any $i, j$ s. t. $i^* < i < j$ they strictly prefers $a_i$ to $a_j$

example: choosing the legal drinking or voting age, the tax rate, the temperature of the office space, the location of a facility along a road etc..

**Proposition:**
if all preferences are single-peaked the median of all individual peaks $a_{i*}$ is the Condorcet winner and defines a strategyproof rule

the family of strategyproof voting rules respecting Anonymity is well understood (Moulin [1980]) and contains for instance the rule selecting the $k$-th ranked peak starting from the left

**Theorem** Gibbard and Satterthwaite (1974)
suppose each voter' preferences can be any linear ordering of $A$
and $A$ contains 3 or more candidates/outcomes

**then**:if a certain voting rule is strategyproof and each outcome will be selected at some profile of ballots
the rule must be **dictatorial**: there is a voter $i^*$ such that the rule always selects the top outcome in $i^*$'s ballot
and vice versa

Merci

Thank You